Week 5: Recall Given $n\times n$ matrix A, the following are equivalent:

① A is non-singular

② A and I_n are row equivalent

③ $N(A) = 0$

④ $\forall b$, $LS(A, b)$ admits a unique sol.

By ②: A and I_n are row equivalent

$\Rightarrow \exists E_1, E_2, \cdots, E_N$ corresponds to row operation s.t.

$$E_N E_{N-1} \cdots E_2 \cdot E_1 \cdot A = I_n.$$

Conversely,

Lemma: Given a $n\times n$ matrix A, if $\exists \underset{n\times n}{B}$ s.t. $BA = I_n$

then A is non-singular.

pf: consider the equ. $Ax = 0$

$$\Rightarrow B\cdot Ax = B\cdot 0 = 0$$

$$\Rightarrow x = 0$$

$\therefore N(A) = \{0\}$ #

Instead of considering $E_N \cdots E_1 \cdot A = I_n$

consider $E_N \cdots E_1 \cdot I_n = B.$

$\Rightarrow$ The matrix $B = E_N \cdots E_1$ is row equivalent to I_n.

$\therefore N(B) = \{0\} \Rightarrow \exists\, G$ s.t. $G \cdot B = I_n$

Q: what is G??

$$\begin{cases} GB = I_n \\ BA = I_n \end{cases} \Rightarrow \quad G = G(BA) = (GB)A = A \quad \cancel{A}$$

$$\therefore \quad BA = AB = I_n$$

Summarise

Lemma: If an $n \times n$ matrix A is non-singular

then $\exists\, B$ s.t. $BA = AB = I_n$.

Eg: $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

$$[A|I_n] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 5 & 8 & 0 & 3 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 & -3 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$\therefore B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & -1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

then $AB = BA = I_3$.

Defn (Recall)

Given a $n \times n$ matrix A, we say that A is invertible

If $\exists \overset{(n\times n)}{B}$ s.t. $AB = BA = I_n$.

B is said to be the inverse of A, denote by A^{-1}.

Eg: $\cdot$ $I_n^{-1} = I_n$

$\cdot$ O_n is Not invertible.

Properties of inverse:

Lemma: If A, B are $n\times n$ matrix which are invertible then AB is also invertible with inverse $= B^{-1}A^{-1}$!

pf: $\cdot$ $B^{-1}A^{-1}(AB) = B^{-1}B = I_n$

$\cdot$ $(AB)B^{-1}A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$

$\therefore$ $\exists$ inverse which is given by $B^{-1}A^{-1}$.

Lemma: If A is invertible, then the inverse is unique.

pf: $\begin{cases} AB = A\tilde{B} = I_n \\ BA = \tilde{B}A = I_n \end{cases}$

then $B = BA\tilde{B} = \tilde{B}$ $\#$

Rmk: that is why A^{-1} is well-defined notion

Lemma: If A is invertible, then A^{-1} is also invertible

Pf: $A^{-1}A = AA^{-1} = I_n$

$\Rightarrow$ A = inverse of A^{-1} $\left((A^{-1})^{-1} = A \right)$

Lemma: Suppose A = $n \times n$ matrix, then A is non-singular iff A is invertible.

Pf: $(\Leftarrow)$: since A^{-1} exists,

$$Ax = 0 \Rightarrow x = A^{-1}0 = 0 \quad \#$$

$(\Rightarrow)$: A is row equivalent to I_n

$\Rightarrow$ $BA = I_n$ for some B

and B is row equivalent to I_n

$\Rightarrow$ $B = A^{-1}$.

☆☆☆ The Hidden important fact:

Lemma: Every row operation matrix is invertible !!
(equivalently, the row operation is reversible!)

Illustration:

① $\lambda R_k + R_i$, $\lambda \neq 0$

② λR_i, $\lambda \neq 0$

③ $R_i \leftrightarrow R_j$

Reverse ②, ③ is simple

② $^{-1}$: $\lambda^{-1} R_i$ ③ $^{-1}$: $R_i \leftrightarrow R_j$

① $^{-1}$: New i-th row = $\tilde{R}_i = \lambda R_k + R_i$

$\therefore R_i = \tilde{R}_i + (-\lambda) R_k$.

i.e. old i-th row = adding $(-\lambda)$ multiple of k-th row to the new i-th row.

Also row operation!!

---

* Expectation: Inverse of row operation matrix

= row operation matrix corresponds to the reverse row operation.

---

pf of lemma:

It suffices to consider the single row operation.

① if row operation $= \alpha R_i + R_j$, $\alpha \in \mathbb{R}$

the matrix $= I_n + \alpha E_{j,i}^{nn}$

claim: the inverse $= I_n - \alpha E_{j,i}^{nn}$ (corresponds to $-\alpha R_i + R_j$)

$$(I_n + \alpha E_{j,i}^{nn})(I_n - \alpha E_{j,i}^{nn})$$

$$= I_n + \cancel{\alpha E_{j,i}^{nn}} - \cancel{\alpha E_{j,i}^{nn}} + \alpha^2 (E_{j,i}^{nn})^2$$

$$= I_n + \alpha^2 (E_{j,i}^{nn})^2.$$

where $\left(\left(E^{nn}_{ji}\right)^2\right)_{kl} = \sum_{m=1}^{n} \left(E^{nn}_{ji}\right)_{km} \left(E^{nn}_{ji}\right)_{ml}$ only take value when $(k=j,\ i=m)$ when $(j=m,\ i=l)$

$$= \sum_{m=1}^{n} \delta_{jk}\delta_{im}\cdot\delta_{jm}\delta_{il}$$

$$= \delta_{jk}\delta_{ij}\delta_{il} = 0 \quad (\text{since } i \neq j) \ \#$$

$\therefore$ Product $= I_n$.

Similarly, $\left(I_n - \alpha E^{nn}_{ji}\right)\left(I_n + \alpha E^{nn}_{ji}\right) = I_n$ #

② : If row operation : λR_i, $\lambda \neq 0$.

then matrix $= I_n + (\lambda - 1) E^{nn}_{ii}$

Claim : Inverse $= I_n + (\lambda^{-1} - 1) E^{nn}_{ii}$

pf: $\left(I_n + (\lambda-1) E^{nn}_{ii}\right)\left(I_n + (\lambda^{-1}-1) E^{nn}_{ii}\right)$

$$= I_n + (\lambda-1) E^{nn}_{ii} + (\lambda^{-1}-1) E^{nn}_{ii} + (\lambda-1)(\lambda^{-1}-1)\left(E^{nn}_{ii}\right)^2$$

$$= I_n + (\lambda+\lambda^{-1}-2) E^{nn}_{ii} + \underbrace{(\lambda-1)(\lambda^{-1}-1)\left(E^{nn}_{ii}\right)^2}_{☆ = 0\ ??}$$

$$\left(E^{nn}_{ii}\right)^2_{kl} = \sum_{m=1}^{n} \left(E^{nn}_{ii}\right)_{km}\left(E^{nn}_{ii}\right)_{ml}$$

$$= \sum_{m=1}^{n} \delta_{ik}\delta_{im}\delta_{im}\delta_{il} = \begin{cases} 0 & \text{if } k \neq i \text{ or } i \neq l \\ 1 & \text{if } k = i = l \end{cases}$$

$\therefore \quad (E^{nn}_{ii})^2 = E^{nn}_{ii}$

In fact, $E^{mn}_{ij} E^{nn}_{kl} = E^{mn}_{il}$ if $k=j$
$= 0$ otherwise

$\Rightarrow \star = 0$ #

Similarly, $(I_n + (\lambda^{-1}-1)E^{nn}_{ii})(I_n + (\lambda - 1)E^{nn}_{ii}) = I_n$ (or replace λ by λ^{-1}) #

---

③ : $R_i \leftrightarrow R_j$

$$\text{Matrix} = I_n - E^{nn}_{ii} + E^{nn}_{ij} + E^{nn}_{ji} - E^{nn}_{jj}$$

Claim: Inverse = same matrix.

i.e. $(I_n - E^{nn}_{ii} + E^{nn}_{ij} + E^{nn}_{ji} - E^{nn}_{jj})^2 = I_n$??

pf: $(I_n - E^{nn}_{ii} + E^{nn}_{ij} + E^{nn}_{ji} - E^{nn}_{jj})^2$

$$= - E^{nn}_{ii}(\cancel{I_n}^{1} - \cancel{E^{nn}_{ii}}^{1} + \cancel{E^{nn}_{ij}}^{2} + \cancel{E^{nn}_{ji}} - \cancel{E^{nn}_{jj}})$$

using $E^{mn}_{ij} E^{nn}_{ii} = 0$

$$+ E^{nn}_{ij}(\cancel{I_n}^{2} - \cancel{E^{nn}_{ii}} + \cancel{E^{nn}_{ij}} + \cancel{E^{nn}_{ji}}^{5} - \cancel{E^{nn}_{jj}}^{8})$$

$$+ E^{nn}_{ji}(\cancel{I_n}^{3} - \cancel{E^{nn}_{ii}}^{3} + \cancel{E^{nn}_{ij}}^{6} + \cancel{E^{nn}_{ji}} - \cancel{E^{nn}_{jj}})$$

$$- E^{nn}_{jj}(\cancel{I_n}^{4} - \cancel{E^{nn}_{ii}} + \cancel{E^{nn}_{ij}} + \cancel{E^{nn}_{ji}}^{7} - \cancel{E^{nn}_{jj}}^{9})$$

$$+ I_n(I_n - \cancel{E^{nn}_{ii}}^{5} + \cancel{E^{nn}_{ij}}^{8} + \cancel{E^{nn}_{ji}}^{7} - \cancel{E^{nn}_{jj}}^{6})$$

$= I_n$ #

Corollary: Given a $n \times n$ matrix A,

The following is equivalent.

① A is invertible

② A is non-singular

③ $\exists$ $n \times n$ matrix H s.t. $HA = I_n$

④ $\exists$ $n \times n$ matrix B s.t. $AB = I_n$

pf: ① $\Leftrightarrow$ ② done before.

① $\Rightarrow$ ③ done before

③ $\Rightarrow$ ①: $HA = I_n \Rightarrow$ A is non-singular

$\Rightarrow$ ② $\Rightarrow$ ①

① $\Rightarrow$ ④: done by before

④ $\Rightarrow$ ①: by ③, B is non-singular and invertible

$\Rightarrow$ B^{-1} exists

$\Rightarrow A = B^{-1} \Rightarrow$ A has a inverse given by $(B^{-1})^{-1} = B$. #

Corollary: ① The inverse of any invertible matrix is given by product of finitely many row operation matrix.

② Any non-singular (or invertible) matrix is a product of finitely many row operation matrix.

pf: ① $\Leftrightarrow$ ②. Suffice to show ①:

A = Invertible $\Rightarrow$ $\exists$ row operation matrix $E_1, \cdots, E_N$ s.t.

$$E_1 \cdots E_N A = I_n$$

$\therefore$ $B = E_1 \cdots E_N$ is s.t. $BA = I_n$

$\Rightarrow$ $A^{-1} = B$ (since A is invertible) #.

Lemma: If A, B are $n \times n$ matrix s.t. AB is non-singular, then A, B are non-singular.

pf: consider $Bx = 0$ $\Rightarrow$ $ABx = 0$ $\Rightarrow$ $x = 0$

$\therefore$ $N(B) = \{0\}$ $\therefore$ B is invertible

$$I_n = AB(AB)^{-1} = A \cdot (B \cdot (AB)^{-1})$$

$\therefore$ $\exists \tilde{B}$ s.t. $A\tilde{B} = I_n$ $\Rightarrow$ A is invertible.

OR.

$$A = \underline{\underline{AB}} \cdot \underline{\underline{B^{-1}}}$$

Both invertible and hence invertible.